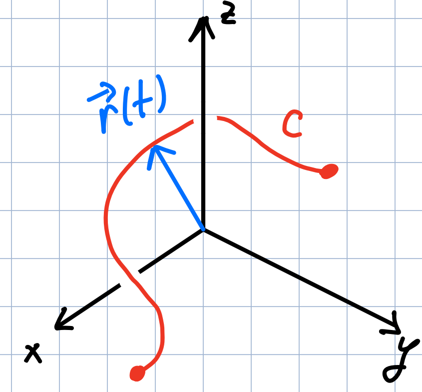
LAST TIME: Fundamental THM of line integrals:

Let C be a curve given by $\vec{r}(t)$, $a \le t \le b$.

Then

$$\boxed{\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).}$$

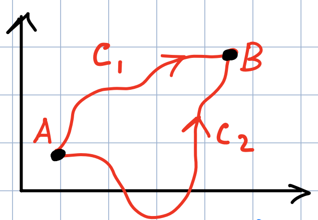


$\vec{F}$ - vector field. The following are equivalent:

(a) $\vec{F}$ is conservative,

i.e. $\vec{F} = \nabla f$

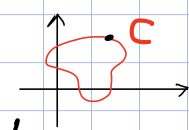
for some function f.

(b) 

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

(c) $\displaystyle\int_C \vec{F} \cdot d\vec{r} = 0$

for any <u>closed</u> curve



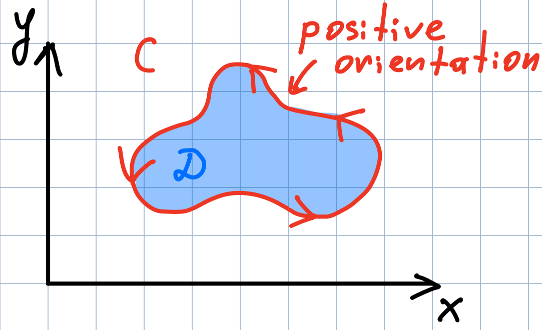
Let $\vec{F}(x,y)$ be a vector field in a simply-connected region D

single-piece, without holes

$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$$

THEN $\vec{F}(x,y)$ is conservative iff $\boxed{\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}}$

Green's theorem



Simple, closed curve
(not crossing itself)

with positive orientation (counterclockwise)
(region is always on the left as we go around C)

bounding the region $\mathcal{D}$.

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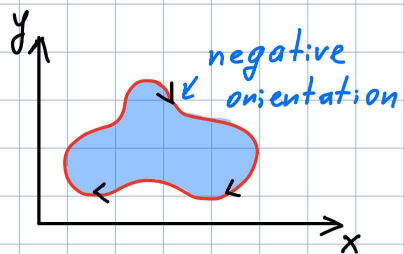

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Green's THM: $P(x,y)$, $Q(x,y)$

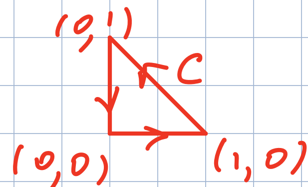
$$\int_C P\,dx + Q\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

positively oriented

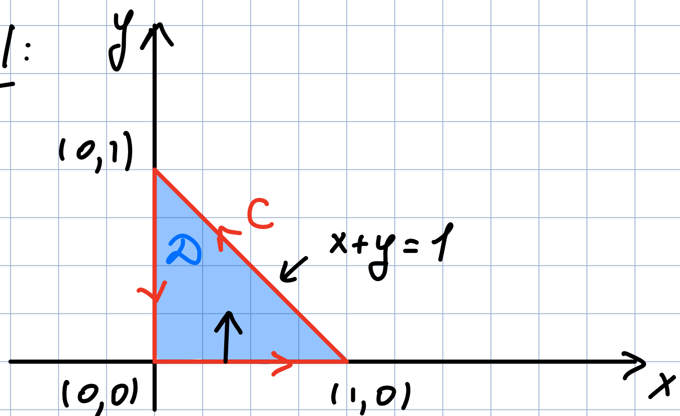
Rmk: One can also use notation $\oint_C P\,dx + Q\,dy$

For a line integral over a closed curve.

Ex: Find the integral $\int_C x^4\,dx + xy\,dy$ over the triangular curve C, using Green's thm.



Sol:



$$\mathcal{D}: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{cases}$$

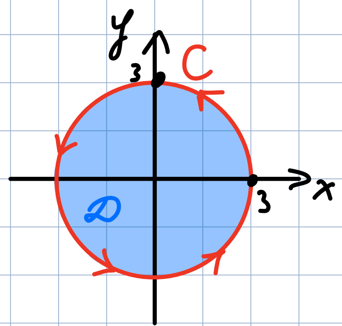
$$\int_C \underbrace{x^4}_{P}\,dx + \underbrace{xy}_{Q}\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_0^1 \underbrace{\int_0^{1-x} (y - 0)\,dy}_{\frac{1}{2}y^2\Big|_{y=0}^{y=1-x} = \frac{(1-x)^2}{2}}\,dx$$

$$= \int_0^1 \frac{1}{2}(1-x)^2\,dx = -\frac{1}{2}\cdot\frac{1}{3}(1-x)^3\Big|_0^1 = \frac{1}{6}$$

Ex: Find $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$, where C is the circle $x^2+y^2=9$.

Sol:



$$\oint_C \underbrace{(3y - e^{\sin x})}_{P}\,dx + \underbrace{(7x + \sqrt{y^4+1})}_{Q}\,dy$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (7 - 3)\,dA$$

$$= 4\int_0^{2\pi}\int_0^3 r\,dr\,d\theta = 4 \cdot 2\pi \cdot \frac{9}{2} = 36\pi$$

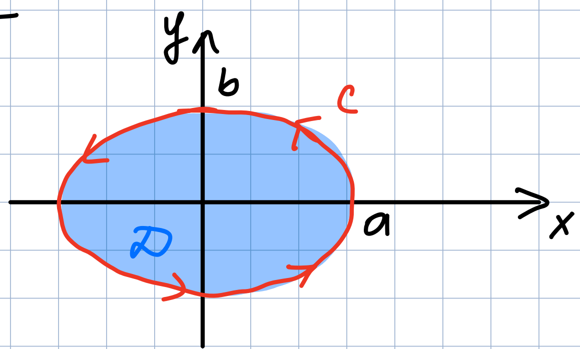
Rmk: Can use Green's thm in the other direction: convert a double integral to a line one.

e.g. $\underbrace{\frac{1}{2}\oint_C \underbrace{x}_{Q}\, dy - \underbrace{y}_{P}\, dx}_{(*)} = \frac{1}{2}\iint_{D} (1-(-1))\, dA = \iint_{D} dA = (\text{Area of } D)$

So, can calculate area as line integral (*) over the boundary.

Ex: D - enclosed by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the area.

Sol:



C: $x = a\cos t$
$y = b\sin t$
$0 \le t \le 2\pi$

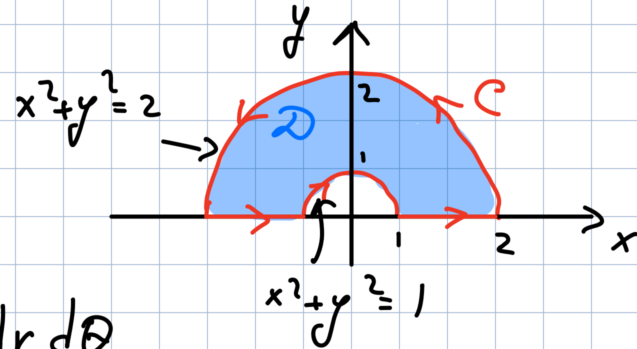
$$\text{area} = \frac{1}{2}\oint_C x\,dy - y\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos t\, y'(t) - b\sin t\, x'(t))\, dt$$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos t\, b\cos t - b\sin t\,(-a\sin t))\, dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2 t + \sin^2 t)\, dt = \frac{1}{2}\int_0^{2\pi} ab\, dt = \pi ab$$

Ex: Find $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semi-annular region:



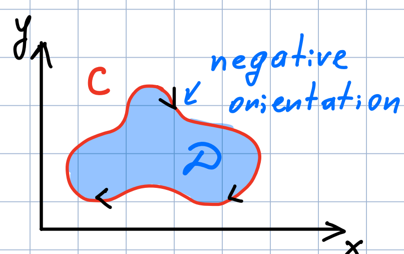
Sol: $\oint_C \underbrace{y^2}_{P}\,dx + \underbrace{3xy}_{Q}\,dy$

$$= \iint_D \underbrace{(3y - 2y)}_{y}\,dA \overset{\text{polar}}{\underset{\text{coord.}}{=\!=}} \int_0^{\pi}\underbrace{\int_1^2 \underbrace{r\sin\theta}_{y}\, r\,dr}\,d\theta$$

$$\sin\theta \, \frac{r^3}{3}\Big|_{r=1}^{r=2} = \frac{7}{3}\sin\theta$$

$$= \int_0^{\pi} \frac{7}{3}\sin\theta\,d\theta = -\frac{7}{3}\cos\theta\Big|_0^{\pi} = \frac{14}{3}$$

Rmk: For a negatively oriented curve C:



$$\int_C P\,dx + Q\,dy = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$